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2003 J. Phys. A: Math. Gen. 36 4651

(http://iopscience.iop.org/0305-4470/36/16/314)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4651–4665 PII: S0305-4470(03)59115-6

On the combinatorics of normal ordering bosonic operators and deformations of it

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Received 3 February 2003, in final form 4 March 2003
Published 8 April 2003
Online at stacks.iop.org/JPhysA/36/4651

Abstract

Recently some combinatorial aspects for the normal ordering of powers of arbitrary monomials of boson operators were discussed. In particular, it was shown that the resulting formulae lead to generalizations of the usual Bell and Stirling numbers. In this paper these considerations are generalized to the q-deformed case. In particular, it is shown that the simplest example of this generalization leads to q-deformed Lah numbers. The connection between (generalized) Stirling and Bell numbers and matrix elements of the above-mentioned operators with respect to the usual Fock space basis and coherent states is discussed.

PACS numbers: 02.10.Ox, 03.65.−w, 05.30.−d

1. Introduction

The process of normal ordering noncommuting operators has been the subject of interest since the beginning of quantum theory. Let us consider a single boson associated with the operators $a, a^\dagger$, satisfying the commutation relation $[a, a^\dagger] = 1$. It has been known for some time that in this case the normal ordering of $(a^\dagger a)^n$ (i.e., moving all the operators a to the right) has a close relation to the Stirling numbers of second kind [1]. Recently, it has been realized that the associated Bell numbers appear when matrix elements of the operators $(a^\dagger a)^n$ are considered with respect to coherent states [2]. These Stirling (and associated Bell) numbers have a long history and have played a major role in combinatorics (see, e.g., [3, 4]). They have been generalized in several ways in the last century. In particular, various q-deformed Stirling numbers have been considered in the mathematical literature in connection with q-analysis and geometry over finite fields (see, e.g., [5–13]). In the physical literature various variants of q-deformed bosons (or bosonic oscillators) have been introduced and studied (see, e.g., [14–28] (for a more complete list of references see the bibliography of [29])). A connection between the normal ordering of q-deformed bosons and q-deformed

Stirling numbers was established in [20] and has been considered further in [2, 30, 31]. Very recently, generalizations of the 'classical' Stirling numbers were introduced in connection with conformal transformations in the plane [32] and with the normal ordering of ordinary bosons, where one considers expressions of the form $[(a^\dagger)^r a^s]^n$ for natural numbers r, s [33–35]. First properties of these generalized Stirling (and associated Bell) numbers have been established in [32–35]. It is, therefore, natural to consider the analogous q-deformed generalized Stirling numbers appearing in the normal ordering of the corresponding expressions for the q-deformed bosons, thus following the same strategy exploited successfully by Katriel and co-workers in the case $r = s = 1$ [2, 20, 30, 31]. This is what we will begin here. Let us note that the Hermitian operator $(a^\dagger)^r a^r$ is of importance in quantum optics (see, e.g., the references given in [33–35]). In recent works the relation between q-deformed oscillators and certain nonlinear (undeformed) oscillators is studied (see, e.g., the literature given in [25]). In the study of the dynamical properties of the q-deformed oscillator and its comparison with the undeformed case the q-deformed Stirling numbers as well as the q-deformed Dobinski relation are used [25]. The approach of introducing nonlinearities into a system by q-deforming it has been applied to various systems and in particular to quantum optics (see, e.g., [27] and the references given therein). Therefore, one should expect that the q-deformation of $(a^\dagger)^r a^r$ should play a role in q-deformed quantum optics (and indeed, it appears, e.g., in [27]). Apart from this application the number operator of the q-deformed oscillator consists of a (weighted) sum over the operators $(a^\dagger)^r a^r$, so that whenever powers of the number operator are considered the q-deformed generalized Stirling (and Bell) numbers will appear. Let us stress that the q-deformed boson we will consider is that considered in [14, 15] (in the literature also called 'math' boson), but we will very briefly discuss its relation to the q-deformed boson of [16, 17] (the 'phys' boson). More general (and systematic) approaches to deformations of bosons (and fermions) have been considered, the most famous of them using quantum groups, see, e.g., [29]. Recently another algebraic approach based on the q-deformed phase space introduced in [36] has been discussed [21–23]. The q-deformed phase space is based on noncommutative variables x, p (as well as a further variable); the associated $*$-algebra is interpreted as the algebra of observables. It is possible to show that the creation and annihilation operators of the q-deformed oscillator can be expressed as particular linear combinations of these generating variables [21–23, 26].

This paper is organized as follows. In section 2 we recall some basic facts about the 'classical' Stirling numbers and some of the results of [32–35] about the generalized Stirling numbers. Some additional properties of the generalized Stirling numbers are derived in section 3. In section 4 we consider the q-deformed boson and the associated q-deformed Stirling numbers, following the above-mentioned strategy of [2, 20, 30, 31]. In section 5 the q-deformed generalized Stirling numbers are discussed and it is shown that they are given in the first non-trivial case by q-deformed Lah numbers. Some conclusions are drawn in section 6.

2. Bosons, Stirling numbers and generalized Stirling numbers

Recall that the standard bosonic commutation relations $[a, a^\dagger] = 1$ can be realized formally in a suitable space of functions by letting $a = \frac{\mathrm{d}}{\mathrm{d}x}$ and $a^\dagger = x$ (operator of multiplication with the identity). When considering the action of $\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n$ on $f(x)$ certain integers $S(n, k)$, the Stirling numbers of second kind [3] appear

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x) = \sum_{k=1}^{n} S(n, k) x^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x). \tag{1}$$

Equation (1) may also be written as

$$(a^\dagger a)^n = \sum_{k=1}^{n} S(n,k)(a^\dagger)^k a^k \tag{2}$$

thereby exemplifying the normal ordering problem of writing $(a^\dagger a)^n$ with all the operators a on the right. Note that there exists a relation

$$S(n+m,k) = \sum_{\mu,\nu=1}^{k} \binom{\mu}{k-\nu} \frac{\nu!}{(k-\mu)!} S(n,\nu)S(m,\mu) \tag{3}$$

reducing for $m = 1$ to the well-known recursion relation

$$S(n+1,k) = S(n,k-1) + kS(n,k) \tag{4}$$

of the Stirling numbers (cf [3], p 226). Using generating functions, it is straightforward to derive from (4) an explicit expression for the Stirling numbers (cf [37], p 19):

$$S(n,k) = \sum_{p=0}^{k} (-1)^{k-p} \frac{p^{n-1}}{(p-1)!(k-p)!} \equiv \frac{(-1)^k}{k!} \sum_{p=0}^{k} (-1)^p \binom{k}{p} p^n. \tag{5}$$

Introducing the falling factorials by $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$, the Stirling numbers can also be written as connection coefficients (cf [4], p 207),

$$x^n = \sum_{k=1}^{n} S(n,k) x^{\underline{k}}. \tag{6}$$

Denoting the rising factorials by $x^{\bar{n}} = x(x+1)\cdots(x+n-1)$, one may also define the Stirling numbers of first kind by $x^{\bar{n}} = \sum_{k=1}^{n} s(n,k)x^k$. In the following we will consider only Stirling numbers of second kind. $S(n,k)$ has the combinatorial interpretation of counting the number of partitions of a set of n distinguishable elements into k non-empty sets. The Bell numbers $B(n)$ are defined by

$$B(n) = \sum_{k=1}^{n} S(n,k) \tag{7}$$

(and $B(0) = 1$, $S(n,0) = \delta_{n,0}$ by convention). Inserting $f(x) = \mathrm{e}^x$ in (1), using the series expansion of the exponential function and evaluating the resulting expression at $x = 1$ yields the Dobinski relation (cf [4], p 210)

$$B(n) = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{k!}. \tag{8}$$

To show a connection to coherent states, we first recall that the harmonic oscillator has Hamiltonian $H = a^\dagger a$ (neglecting the zero-point energy) and the usual eigenstates $|n\rangle$ (for $n \in \mathbf{N}$) satisfying $H|n\rangle = n|n\rangle$ and $\langle m|n\rangle = \delta_{mn}$. Let us now define for $z \in \mathbf{C}$ the coherent state

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \tag{9}$$

These states are normalized, i.e., $\langle z|z\rangle = 1$, and satisfy $a|z\rangle = z|z\rangle$. It has been noted in [2] that for z with $|z|^2 = 1$ one has (due to (2) and (7))

$$\langle z|(a^\dagger a)^n|z\rangle = \sum_{k=1}^{n} S(n,k) = B(n). \tag{10}$$

More generally, one may consider the matrix elements of $e^{\lambda a^\dagger a}$. Following [2] we introduce $f_z(\lambda) \equiv \langle z|e^{\lambda a^\dagger a}|z\rangle$ and take a derivative with respect to λ, obtaining $\frac{\mathrm{d}f_z(\lambda)}{\mathrm{d}\lambda} = \langle z|e^{\lambda a^\dagger a}a^\dagger a|z\rangle$. Using $e^{\lambda a^\dagger a}a^\dagger a = a^\dagger e^{\lambda(a^\dagger a+1)}a$, this leads to the initial-value problem

$$\frac{\mathrm{d}f_z(\lambda)}{\mathrm{d}\lambda} = |z|^2\,e^{\lambda}\,f_z(\lambda) \qquad f_z(0) = 1. \tag{11}$$

The solution is given by $f_z(\lambda) = \exp(|z|^2[e^{\lambda} - 1])$. From (10) it follows also that for $|z|^2 = 1$ one has $f_z(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}B(n)$. This implies the well-known exponential generating function of the Bell numbers (cf [3, 4])

$$e^{e^{\lambda}-1} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}B(n). \tag{12}$$

The authors of [33–35] introduced for $r \geqslant s$ certain generalized Stirling numbers $S_{r,s}(n,k)$ by

$$[(a^\dagger)^r a^s]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^\dagger)^k a^k \tag{13}$$

clearly $S_{1,1}(n,k) \equiv S(n,k)$ from above. The case of $s = 1$ was considered in [32] in connection with conformal transformations on the plane. In fact, the vector fields $\left\{x^r \frac{\mathrm{d}}{\mathrm{d}x}\right\}_{r\geqslant 0}$ are part of the classical Virasoro algebra (no central charge). In [33–35] generalized Bell numbers $B_{r,s}(n)$ (with $B_{1,1}(n) \equiv B(n)$ from above) were defined as

$$B_{r,s}(n) = \sum_{k=s}^{ns} S_{r,s}(n,k) \tag{14}$$

and explicit formulae were given for $B_{r,s}(n)$ and $S_{r,s}(n,k)$; in particular,

$$S_{r,r}(n,k) = \frac{(-1)^k}{k!}\sum_{p=0}^{k}(-1)^p \binom{k}{p}(p^{\underline{r}})^n. \tag{15}$$

Furthermore, the generalized Bell numbers were interpreted as moments of positive measures and equation (10) was generalized to the case of arbitrary r and s, yielding $\langle z|[(a^\dagger)^r a^s]^n|z\rangle = B_{r,s}(n)$.

3. Further properties of generalized Stirling numbers

Relation (4) may be shown directly using

$$a(a^\dagger)^r = (a^\dagger)^r a + r(a^\dagger)^{r-1} \tag{16}$$

for $r = n + 1$. Iterating (16), one obtains

$$a^n(a^\dagger)^r = \sum_{k=0}^{n}\binom{n}{k}\frac{r!}{(r-n+k)!}(a^\dagger)^{r-n+k}a^k. \tag{17}$$

Writing $(a^\dagger a)^{n+m} = (a^\dagger a)^n(a^\dagger a)^m$, using on both sides (2) and commuting on the right-hand side all a to the right with the help of (17) yields (3). The same strategy may be applied for the expression $[(a^\dagger)^r a^s]^{n+m}$. The result is

$$S_{r,s}(n+m,k) = \sum_{\mu,\nu=s}^{k}\binom{\mu}{k-\nu}\frac{\{n(r-s)+\nu\}!}{\{n(r-s)+k-\mu\}!}S_{r,s}(n,\nu)S_{r,s}(m,\mu). \tag{18}$$

Note that the formula simplifies drastically in the case $r = s$ (and reduces for $r = 1 = s$ to (3)). From (18) one obtains for $m = 1$ the recursion relation

$$S_{r,s}(n+1,k) = \sum_{\nu=k-s}^{k} \binom{s}{k-\nu} \frac{\{n(r-s)+\nu\}!}{\{n(r-s)+k-s\}!} S_{r,s}(n,\nu) \tag{19}$$

which generalizes (4) to the case of arbitrary r and s. In the case $r = s$ the relation (19) has already been given in [35]. In analogy with (6) these generalized Stirling numbers are also connection coefficients [35]:

$$(x^{\underline{r}})^n = \sum_{k=r}^{rn} S_{r,r}(n,k) x^{\underline{k}}. \tag{20}$$

Considering (19) for $s = 1$, only two terms remain on the right-hand side, yielding

$$S_{r,1}(n+1,k) = S_{r,1}(n,k-1) + \{n(r-1)+k\} S_{r,1}(n,k). \tag{21}$$

Relation (21) appeared already in [32]. Choosing $r = 2$ yields

$$S_{2,1}(n+1,k) = S_{2,1}(n,k-1) + (n+k) S_{2,1}(n,k) \tag{22}$$

which is exactly the defining relation for the (signless) Lah numbers $L(n,k)$ [33, 34]; thus, one obtains the explicit expression

$$S_{2,1}(n,k) = L(n,k) \equiv \frac{n!}{k!} \binom{n-1}{k-1}. \tag{23}$$

The signless Lah numbers are the connection coefficients between rising and falling factorials (cf [4], p 156),

$$x^{\bar{n}} = \sum_{k=0}^{n} L(n,k) x^{\underline{k}}. \tag{24}$$

Let us now derive a generalized Dobinski relation for the $B_{r,r}(n)$ following the approach sketched in section 2 for $B(n) \equiv B_{1,1}(n)$. First recall that we can write (13) in the case $r = s$ also as

$$\left[x^r \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^r \right]^n f(x) = \sum_{k=r}^{nr} S_{r,r}(n,k) x^k \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^k f(x). \tag{25}$$

Choosing $f(x) = \mathrm{e}^x$, one checks that

$$x^r \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^r \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=r}^{\infty} \frac{k^{\underline{r}} x^k}{k!}.$$

Iterating this n times and inserting the result in (25) yields

$$\sum_{k=r}^{\infty} \frac{(k^{\underline{r}})^n x^k}{k!} = \sum_{k=r}^{nr} S_{r,r}(n,k) x^k \, \mathrm{e}^x.$$

Now, choosing $x = 1$ and recalling (14) gives the generalized Dobinski relation

$$B_{r,r}(n) = \frac{1}{e} \sum_{k=r}^{\infty} \frac{(k^{\underline{r}})^n}{k!} \tag{26}$$

which reduces for $r = 1$ to the Dobinski relation (8). Relation (26) was stated also in [35] (and in a slightly different form also in [33, 34]). Let us consider the exponential generating function $\sum_n \frac{\lambda^n}{n!} B_{r,r}(n)$. Inserting the explicit expression for $B_{r,r}(n)$ (and the convention $B_{r,r}(0) = 1$), a careful calculation shows that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} B_{r,r}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{\mathrm{e}^{\lambda k^{\underline{r}}}}{k!} - \frac{1}{e} \sum_{k=0}^{r-1} \frac{\mathrm{e}^{\lambda k^{\underline{r}}} - 1}{k!}.$$

This gives an alternative expression to that given in [35]. In particular, the second sum on the right-hand side vanishes for $r = 1$ and the first one gives $\mathrm{e}^{\mathrm{e}^{\lambda}-1}$, thus reproducing (12). Since one has for $|z|^2 = 1$ that $\langle z|[(a^\dagger)^r a^r]^n|z\rangle = B_{r,r}(n)$, the exponential generating function equals $\langle z|\mathrm{e}^{\lambda(a^\dagger)^r a^r}|z\rangle$, so that the above equation can also be written as (still assuming $|z|^2 = 1$)

$$\langle z|\mathrm{e}^{\lambda(a^\dagger)^r a^r}|z\rangle = \frac{1}{e} \sum_{k=0}^{\infty} \frac{\mathrm{e}^{\lambda k^{\underline{r}}}}{k!} - \frac{1}{e} \sum_{k=0}^{r-1} \frac{\mathrm{e}^{\lambda k^{\underline{r}}} - 1}{k!}.$$

4. Normal ordering of q-deformed bosons/fermions and q-Stirling numbers

Now, we want to consider the q-deformed version of the above story. Recall that the Q-deformed harmonic oscillator was defined in [16, 17] in terms of creation and annihilation operators $b^\dagger$ and b and the number operator N satisfying

$$bb^\dagger - Qb^\dagger b = Q^{-N} \qquad [N, b^\dagger] = b^\dagger \qquad [N, b] = -b \tag{27}$$

here Q is a real number. Clearly, the limit $Q \to 1$ gives the usual commutation relations. (In the following we will abbreviate this limit by writing '$Q = 1$'.) A different version of the deformed harmonic oscillator can be obtained by defining [18, 19] operators $a, a^\dagger$ through equations $b \equiv Q^{\frac{1}{2}} a Q^{-\frac{N}{2}}$ and $b^\dagger \equiv Q^{\frac{1}{2}} Q^{-\frac{N}{2}} a^\dagger$. From (27) one concludes that $[N, a^\dagger] = a^\dagger$, $[N, a] = -a$ as well as

$$aa^\dagger - qa^\dagger a = 1 \tag{28}$$

here we have set $q \equiv Q^2$. This version of the deformed harmonic oscillator has been considered first in [14, 15]. In the following, we will be interested in the combinatorial consequences of (28) (and not in the number operator). Mostly we will treat q as a formal indeterminate (commuting with $a, a^\dagger$), but whenever concrete values are considered, we assume that $q \in (-1, 1]$ (note that the q coming from (27) is positive); the degenerate case $q = 0$ will not be considered explicitly. For $q < 0$ we may write $q \equiv -\tilde{q}$ with a positive $\tilde{q}$; then the relation (28) can be written (with f instead of a) as

$$ff^\dagger + \tilde{q} f^\dagger f = 1 \qquad [N, f^\dagger] = f^\dagger \qquad [N, f] = -f. \tag{29}$$

These are exactly the commutation relations of the $\tilde{q}$-deformed fermionic oscillator as introduced in [38, 39] (in fact, one has the relations $gg^\dagger + Qg^\dagger g = Q^{-N}$, $[N, g^\dagger] = g^\dagger$, $[N, g] = -g$, but the transformations $g \equiv Q^{-\frac{N}{2}} f$, $g^\dagger \equiv f^\dagger Q^{-\frac{N}{2}}$ yield (29) with $\tilde{q} \equiv Q^2$ [39]). Again, considering $\tilde{q} \to 1$ yields the ordinary fermionic oscillator (but see below). Recall that the very similar relations $gg^\dagger + Qg^\dagger g = Q^N$, $[N, g^\dagger] = g^\dagger$, $[N, g] = -g$, $g^2 = (g^\dagger)^2 = 0$ are equivalent to the usual fermionic commutation relations (see, e.g., [38–41]). Thus, considering $q \in (-1, 1]$ in (28) allows us to treat the bosonic and fermionic cases uniformly. Denoting by D_q the q-derivative introduced by Jackson (for q-analysis one may consider, e.g., [42]), i.e.,

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \tag{30}$$

we obtain formally a realization of (28) on a suitable space of functions by letting $a = D_q$ and $a^\dagger = x$, since $[D_q, x]_q = 1$. For $q < 0$ we write $q \equiv -\tilde{q}$, so that $D_q \equiv D_{\tilde{q}}^F$ with

$D^F_{\tilde q} f(x) = \{f(x) - f(-\tilde q x)\}/\{(1+\tilde q)x\}$. In particular, in the limit $\tilde q \to 1$ one finds $D^F_1 f(x) = \{f(x) - f(-x)\}/2x$. Note that $D^F_1 f$ vanishes for symmetric functions (i.e., $f(-x) = f(x)$) and yields $f(x)/x$ for odd functions (i.e., $f(-x) = -f(x)$). Let us intruduce the q-deformed Stirling numbers $S(n,k|q)$ in close analogy with (1) by

$$(xD_q)^n f(x) = \sum_{k=1}^{n} S(n,k|q) x^k D_q^k f(x) \tag{31}$$

equivalently, this may be defined in analogy with (2) by [20]

$$(a^\dagger a)^n = \sum_{k=1}^{n} S(n,k|q)(a^\dagger)^k a^k \tag{32}$$

where $a, a^\dagger$ satisfy (28). The associated q-deformed Bell numbers are given by

$$B(n|q) = \sum_{k=1}^{n} S(n,k|q). \tag{33}$$

It is clear that considering $q = 1$ yields the 'classical' Stirling and Bell numbers, i.e., $S(n,k|q=1) = S(n,k)$ and $B(n|q=1) = B(n)$. It is very tempting to consider also the 'fermionic limit' $q \to -1$ and introduce the 'fermionic' Stirling and Bell numbers $S_F(n,k)$ and $B_F(n)$ by

$$S_F(n,k) = \lim_{q\to -1} S(n,k|q) \qquad B_F(n) = \lim_{q\to -1} B(n|q).$$

Note that $S(n,k|q)$ will be a polynomial in q of degree at most $\frac{n}{2}(n-1)$ (the highest power of q appears when one commutes all a to the right; here one acquires $(n-1)+(n-2)+\cdots+1$ factors of q). Let us give some examples. For $n = 1$ there exists only $S(1,1|q) = 1$; for $n = 2$ one has $S(2,1|q) = 1$ and $S(2,2|q) = q$. More interesting is the case $n = 3$, where one has $S(3,1|q) = 1$, $S(3,2|q) = 2q + q^2$, $S(3,3|q) = q^3$, and the case $n = 4$, where one has $S(4,1|q) = 1$, $S(4,2|q) = 3q + 3q^2 + q^3$, $S(4,3|q) = 3q^3 + 2q^4 + q^5$, $S(4,4|q) = q^6$. Of course, considering $q = 1$ yields the classical Stirling numbers. Considering $q \to -1$ yields $S_F(1,1) = 1$; $S_F(2,1) = 1$, $S_F(2,2) = -1$; $S_F(3,1) = 1$, $S_F(3,2) = -1$, $S_F(3,3) = -1$ as well as $S_F(4,1) = 1$, $S_F(4,2) = -1$, $S_F(4,3) = -1$, $S_F(4,4) = 1$. From the usual fermionic algebra $ff^\dagger + f^\dagger f = 0$ one infers that $(f^\dagger f)^n = f^\dagger f$, so that the 'physical' coefficients $S_F^{\mathrm{ph}}(n,k)$ in $(f^\dagger f)^n = \sum_{k=1}^n S_F^{\mathrm{ph}}(n,k)(f^\dagger)^k f^k$ should be given by $S_F^{\mathrm{ph}}(n,k) = \delta_{k,1}$. Clearly, $S_F(n,k) \neq S_F^{\mathrm{ph}}(n,k)$. But note the ambiguity in the 'definition' of $S_F^{\mathrm{ph}}(n,k)$; since $f^k = 0$ for $k \geqslant 2$, the coefficients $S_F^{\mathrm{ph}}(n,k)$ for $k \geqslant 2$ are in fact arbitrary (as long as they are finite). So, we may conclude that the $S_F(n,k)$ are indeed the fermionic counterpart to the 'bosonic' Stirling numbers $S(n,k)$. This ambiguity when considering the limit $\tilde q \to 1$ in the fermionic situation is closely related to the 'weak exclusion principle' discussed (e.g., in [39, 43, 44]). For the following we introduce the standard notation $[n] \equiv [n]_q = (1+q+\cdots+q^{n-1}) = \frac{1-q^n}{1-q}$ and

$$[n]! \equiv [n][n-1]\cdots[2][1] \qquad [n;k] \equiv \frac{[n]}{[n-k][k]} \tag{34}$$

for the q-factorials and q-binomial coefficients. Note that we will suppress the index q in the following as far as possible, only displaying it when necessary. Let us point out that in the case $q < 0$ we may write $q \equiv -\tilde q$ with a positive $\tilde q$. It follows that

$$[n]_q \equiv \frac{1-q^n}{1-q} = \frac{1-(-\tilde q)^n}{1+\tilde q} \equiv [n]^F_{\tilde q}$$

here the last equation is the definition of the $\tilde{q}$-fermionic basic number appearing in recent studies of the $\tilde{q}$-deformed fermionic oscillator (see, e.g., [38, 39, 43, 44]). Note that the limit $\tilde{q} \to 1$ yields $[n]_{q=-1} \equiv [n]^F_{\tilde{q}=1} = \{1-(-1)^n\}/2$, i.e., zero if n is even and 1 otherwise. In particular, $[n]_{q=-1}! = 0$ for $n \geqslant 2$. Let us now introduce the q-deformed exponential function by

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}$$

where the sum converges uniformly for $|x| < R_q \equiv (1-q)^{-1}$; note that $R_1 = \infty$ whereas taking the limit $q \to -1$ in $e_q(x)$ is problematic. Since $D_q e_q(x) = e_q(x)$, one may apply (31) to $f(x) = e_q(x)$ to obtain (using the series expansion of e_q and $D_q x^m = [m]x^{m-1}$)

$$\frac{1}{e_q(x)} \sum_{k=1}^{\infty} \frac{[k]^n}{[k]!} x^k = \sum_{k=1}^{n} S(n,k|q) x^k$$

which reduces for $x = 1$ to the q-deformed version of the Dobinski relation (8), i.e.,

$$B(n|q) = \frac{1}{e_q(1)} \sum_{k=1}^{\infty} \frac{[k]^n}{[k]!}. \tag{35}$$

This q-deformed Dobinski relation was established by Milne [8] and has been proved by different methods in different contexts (see, e.g., [2, 12, 13]). Iterating relation (28) yields $a(a^\dagger)^n - q^n (a^\dagger)^n a = [n](a^\dagger)^{n-1}$, which may be used to obtain the recursion relation

$$S(n+1,k|q) = q^{k-1} S(n,k-1|q) + [k] S(n,k|q). \tag{36}$$

This is the q-deformed version of the recursion relation of the Stirling numbers, yielding in the limit $q \to 1$ equation (4). Formula (36) shows that the Stirling numbers $S(n,k|q)$ are indeed the q-deformed Stirling numbers in the version considered by Milne [8]. The q-deformed Stirling numbers were introduced in a slightly different form by Carlitz [5, 6] and have been considered (in one or the other version) from a purely mathematical point of view [7–13]. Katriel and Kibler observed [20] that they play an important role in the study of the q-boson. Note that one obtains from (36) in the 'fermionic' limit $q \to -1$ the recursion relation

$$S_F(n+1,k) = (-1)^{k-1} S_F(n,k-1) + \frac{\{1-(-1)^k\}}{2} S_F(n,k) \tag{37}$$

(with initial values $S_F(1,1) = 1$ and $S_F(1,0) = 0$). From (36) one concludes that $S(n+1,n+1|q) = q^n S(n,n|q)$ and since $S(1,1|q) = 1$ that

$$S(n,n|q) = q^{\frac{n}{2}(n-1)} \equiv q^{\binom{n}{2}}.$$

Let us consider two special cases of $S(n,k|q)$. The first case is $k = 2$. From the explicit formula (39)—or directly from (36)—one finds that $S(n,2|q) = [2]^{n-1} - 1$, generalizing the well-known formula $S(n,2) = 2^{n-1} - 1$. The second case is $k = n$. The recursion relation (36) yields $S(n+1,n|q) = q^{\binom{n}{2}}[n] + q^{n-1} S(n,n-1|q)$, from which one infers the analogue of the well-known formula[1] $S(n+1,n) = \frac{n}{2}(n+1)$:

$$S(n+1,n|q) = q^{\binom{n}{2}} \sum_{k=1}^{n} [k].$$

[1] In the undeformed case one may deduce this also from $S(n+1,k+1) = \sum_{l=k}^{n} \binom{n}{l} S(l,k)$ by considering $k = n-1$; according to [31] there exists the following q-deformed analogue of this relation (identity 1 of [31]): $S(n+1,k+1|q) = \sum_{l=k}^{n} \binom{n}{l} S(l,k|q) q^l$. Unfortunately, choosing $k = n-1$ yields $S(n+1,n|q) = q^{\binom{n}{2}} n + q^n S(n,n-1|q)$, whereas choosing $k = n$ in (36) yields $S(n+1,n|q) = q^{\binom{n}{2}}[n] + q^{n-1} S(n,n-1|q)$, indicating that this q-deformed analogue might be incorrect.

Although the general expression (39) is well known in the existing literature, we now sketch how one may determine it in a direct approach following closely the one of the undeformed case. For this we introduce the formal generating function $T_k(x) = \sum_{n\geqslant 0} S(n,k|q)x^n$ (here and in the following calculation we assume that $q \in (-1, 1])$. Now, (36) implies for the generating function

$$T_k(x) = q^{\binom{k}{2}} x^k \prod_{p=1}^{k} \frac{1}{(1-[p]x)}. \tag{38}$$

As in the undeformed case (cf [37], p 19) one writes the product as $\prod_{p=1}^{k} \frac{1}{(1-[p]x)} = \sum_{p=1}^{k} \frac{\alpha_p}{1-[p]x}$. To determine the coefficient α_p, multiply both sides with $(1-[p]x)$ and consider then $x = 1/[p]$. Using $[m]-[n] = q^n[m-n]$, one finds

$$\alpha_p = (-1)^{k-p} \frac{q^{p^2-kp-\binom{p}{2}}[p]^{k-1}}{[p-1]![k-p]!}$$

and therefore

$$T_k(x) = q^{\binom{k}{2}} x^k \sum_{p=1}^{k} (-1)^{k-p} q^{p^2-kp-\binom{p}{2}} \frac{[p]^{k-1}}{[p-1]![k-p]!(1-[p]x)}.$$

Since $S(n,k|q)$ is the coefficient of x^n in $T_k(x)$, this yields the explicit expression

$$S(n,k|q) = \sum_{p=1}^{k} (-1)^{k-p} q^{\binom{k-p}{2}} \frac{[p]^{n-1}}{[p-1]![k-p]!} \equiv \frac{(-1)^k}{[k]!} \sum_{p=0}^{k} (-1)^p q^{\binom{k-p}{2}} [k;p][p]^n \tag{39}$$

which generalizes (5) in a beautiful manner (in the sense that taking $q \to 1$ reproduces (5)). Note that this approach does not work for $q = -1$, since the recursion relation (37) for the $S_F(n,k)$ does not imply the generating function (38). Furthermore, it also seems to be non-trivial to consider the limit $q \to -1$ in the explicit formula (39) to obtain somehow $S_F(n,k)$, since there will appear summands diverging due to $[m]_{q=-1}! = 0$ for $m \geqslant 2$. Thus, one should start directly from relation (37) to obtain an explicit formula. The q-deformed Stirling numbers are also connection coefficients analogous to (6),

$$[x]^n = \sum_{k=1}^{n} S(n,k|q)[x]^{\underline{k}} \tag{40}$$

where $[x]^{\underline{k}} = [x][x-1]\cdots[x-k+1]$. For the following we need the analogue of (17). A slightly tedious induction shows that

$$a^n(a^\dagger)^r = \sum_{k=0}^{n} q^{k(k+r-n)}[n;k] \frac{[r]!}{[r-n+k]!} (a^\dagger)^{r-n+k} a^k. \tag{41}$$

As in the case $q = 1$ this may be used to derive a recursion relation for the $S(n,k|q)$. The result is

$$S(n+m,k|q) = \sum_{\mu,\nu=1}^{k} q^{(k-\mu)(k-\nu)}[\mu;k-\nu] \frac{[\nu]!}{[k-\mu]!} S(m,\mu|q)S(n,\nu|q). \tag{42}$$

This reduces for $q = 1$ to (3) and for $m = 1$ to (36). Let us now discuss the relation of the q-deformed Bell numbers to coherent states; for the following we assume that $q \in (0, 1]$ (the fermionic case with $q \in (-1, 0)$ requires the introduction of 'pseudo-grassmann' variables [39]). Recall that the Fock space associated with (28) is spanned by states $|n;q\rangle$

(with $n = 0, 1, 2, \ldots$) defined by $a|0; q\rangle = 0$ and $|n; q\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0; q\rangle$. The action of the operators $a^\dagger, a$ is given by

$$a^\dagger|n; q\rangle = \sqrt{[n+1]}|n+1; q\rangle \qquad a|n; q\rangle = \sqrt{[n]}|n-1; q\rangle. \tag{43}$$

In this setting one introduces in analogy with (9) the coherent states

$$|z; q\rangle = \{e_q(|z|^2)\}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n; q\rangle. \tag{44}$$

It is then straightforward to show that for z with $|z|^2 = 1$ one has in analogy with (10) the equation [2]

$$\langle q; z|(a^\dagger a)^n|z; q\rangle = B(n|q). \tag{45}$$

It is also straightforward to consider the matrix elements $\langle q; l|(a^\dagger a)^n|m; q\rangle$. They vanish when $l \neq m$ and are given for $l = m$ by

$$\langle q; m|(a^\dagger a)^n|m; q\rangle = \sum_{k=1}^{\min(n,m)} \frac{[m]!}{[m-k]!} S(n, k|q).$$

Consequently, the matrix elements of $e_q(\lambda a^\dagger a)$ are given by

$$\langle q; m|e_q(\lambda a^\dagger a)|m; q\rangle = \sum_{n=0}^{\infty} \sum_{k=1}^{\min(n,m)} \frac{\lambda^n [m]!}{[n]![m-k]!} S(n, k|q). \tag{46}$$

The expressions for the undeformed case follow by considering $q = 1$. It would be highly desirable to have an explicit expression for 'the' exponential generating function of the q-deformed Bell numbers (analogous to (12)). However, in the deformed setting there are two natural series one may consider [8]:

$$\Phi_q(\lambda) = \sum_{n\geqslant 0} \frac{\lambda^n}{[n]!} B(n|q) \qquad \Psi_q(\lambda) = \sum_{n\geqslant 0} \frac{\lambda^n}{n!} B(n|q).$$

No explicit expression seems to be known for either series. Using $B(n+1|q) = \sum_{k=0}^{n} \binom{n}{k} q^k B(k|q)$, it is shown in [8] that

$$\frac{\mathrm{d}\Psi_q(\lambda)}{\mathrm{d}\lambda} = \mathrm{e}^{\lambda} \Psi_q(q\lambda). \tag{47}$$

For the 'more natural' candidate Φ_q no such simple relation is known. Let us, therefore, try to imitate the approach sketched in section 2 using coherent states and see what the problem is. We introduce the function

$$g_z(\lambda) = \langle q; z|e_q(\lambda a^\dagger a)|z; q\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{[n]!} \sum_{k=0}^{n} S(n, k|q)|z|^{2k}$$

note that for z with $|z|^2 = 1$ one has $g_z(\lambda) = \Phi_q(\lambda)$. Taking the q-derivative with respect to λ gives

$$D_\lambda g_z(\lambda) = \langle q; z|e_q(\lambda a^\dagger a) a^\dagger a|z; q\rangle = |z|^2 \langle q; z|e_q(\lambda[q a^\dagger a + 1])|z; q\rangle$$

where we have used in the second equation that $e_q(\lambda a^\dagger a) a^\dagger a = a^\dagger e_q(\lambda[q a^\dagger a + 1]) a$. Note that if it were true that $e_q(\lambda[q a^\dagger a + 1]) = e_q(\lambda) e_q(\lambda q a^\dagger a)$ we would obtain in analogy with (11) and (47) the equation $D_\lambda \Phi_q(\lambda) = e_q(\lambda) \Phi_q(q\lambda)$. Unfortunately, there does not exist a simple addition law for the q-deformed exponential function (but see [45] and the references given therein for a discussion of addition laws of related q-exponential functions). Let us therefore try the direct approach of inserting the explicit expression for $B(n|q)$ into the generating function (as we did at the end of section 3). Since the mixing of 'ordinary' and q-numbers seems to produce no interesting result, we consider Ψ_q. A simple calculation (considering q-numbers as real numbers, not polynomials in q) yields

$$\Psi_q(\lambda) = \frac{1}{\mathrm{e}_q(1)} \sum_{k=0}^{\infty} \frac{\mathrm{e}^{\lambda[k]}}{[k]!}.$$

Taking the derivative with respect to λ and using $[k+1] = [1] + q[k] = 1 + q[k]$ shows that $\Psi_q(\lambda)$ satisfies indeed (47). Following the above-mentioned strategy not to mix the two types of numbers, one is tempted to introduce $\Xi_q([\lambda]) = \sum_{n\geqslant 0} \frac{[\lambda]^n}{[n]!} B(n|q)$. Inserting $B(n|q)$ gives $\Xi_q([\lambda]) = (1/e_q(1)) \sum_{k\geqslant 0} \frac{e_q([\lambda][k])}{[k]!}$. Due to the lack of 'good' properties of the q-exponential function, this version does not seem to be interesting. Summarizing the above discussion, it seems to be rather difficult to find an appropriate substitute for the exponential generating function of the Bell numbers in the q-deformed case.

5. The q-deformed generalized Stirling numbers

Let us now introduce the q-deformed generalized Stirling numbers $S_{r,s}(n, k|q)$ by the same equations as in (13),

$$[(a^\dagger)^r a^s]^n = (a^\dagger)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k|q)(a^\dagger)^k a^k$$

but where now the operators satisfy (28). Clearly, $S_{1,1}(n, k|q) \equiv S(n, k|q)$ as well as $S_{r,s}(n, k|q = 1) \equiv S_{r,s}(n, k)$. This may also be written in terms of the q-derivative D_q as

$$[x^r (D_q)^s]^n f(x) = x^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k|q) x^k (D_q)^k f(x). \tag{48}$$

As above one may also introduce the corresponding q-deformed generalized Bell numbers,

$$B_{r,s}(n|q) = \sum_{k=s}^{ns} S_{r,s}(n, k|q).$$

Using (41), one derives the following recursion relation for the $S_{r,s}(n, k|q)$:

$$S_{r,s}(n+m, k|q) = \sum_{\mu,\nu=s}^{k} q^{(k-\nu)\{(k-\mu)+n(r-s)\}} \times [\mu; k-\nu] \frac{[n(r-s)+\nu]!}{[n(r-s)+k-\mu]!} S_{r,s}(n, \nu|q) S_{r,s}(m, \mu|q). \tag{49}$$

Let us state some particular cases explicitly. For $m = 1$ one finds

$$S_{r,s}(n+1, k|q) = \sum_{\nu=k-s}^{k} q^{(k-\nu)\{(k-s)+n(r-s)\}} [s; k-\nu] \frac{[n(r-s)+\nu]!}{[n(r-s)+k-s]!} S_{r,s}(n, \nu|q).$$

Choosing furthermore $s = 1$ simplifies this relation to

$$S_{r,1}(n+1, k|q) = q^{(k-1)+n(r-1)} S_{r,1}(n, k-1|q) + [n(r-1)+k] S_{r,1}(n, k|q). \tag{50}$$

This is a q-deformed version of (21). Recall that the (signless) Lah numbers $L(n,k)$ correspond to the case $r=2$ of (21), i.e., $S_{2,1}(n,k)=L(n,k)$. Choosing $r=2$ in (50) gives the recursion relation

$$S_{2,1}(n+1,k|q)=q^{n+k-1}S_{2,1}(n,k-1|q)+[n+k]S_{2,1}(n,k|q) \tag{51}$$

this reduces for $q=1$ to the recursion relation (22) of the 'ordinary' Lah numbers. Let us denote these q-deformed Lah numbers by $L(n,k|q)\equiv S_{2,1}(n,k|q)$. From the recursion relation it follows directly that $L(n,1|q)=[n]!$ and $L(n,n|q)=q^{n(n-1)}$. In general, the solution to (51) is given by

$$L(n,k|q)=q^{k(k-1)}\frac{[n]!}{[k]!}[n-1;k-1] \tag{52}$$

these numbers are exactly the q-deformed Lah numbers introduced in [9, 10] in a combinatorial context. Let us briefly check that these numbers satisfy indeed (51). First one uses $[r;s]=[r-1;s-1]+q^s[r-1;s]$ to obtain

$$L(n+1,k|q)=q^{2(k-1)}\frac{[n+1]}{[k]}L(n,k-1|q)+q^{k-1}[n+1]L(n,k|q).$$

Now, using $[n+1]=q^{n+1-k}[k]+[n+1-k]$ as well as $[k-1]L(n,k|q)=q^{2(k-1)}[n-k+1]/[k]L(n,k-1|q)$ gives the relation

$$L(n+1,k|q)=q^{n+k-1}L(n,k-1|q)+\{[k-1]+q^{k-1}[n+1]\}L(n,k|q)$$

which is, due to $[k-1]+q^{k-1}[n+1]=[n+k]$, exactly (51). It was already mentioned in [10] that the $L(n,k|q)$ introduced above are, in complete analogy with (24), the connection coefficients between rising and falling q-factorials,

$$[x]^{\bar{n}}=\sum_{k=0}^{n}L(n,k|q)[x]^{\underline{k}}.$$

Recently two different types of q-deformed Lah numbers were introduced in connection with lattices and finite geometries [13] (of course, in the limit $q\to 1$ all of these deformed Lah numbers reproduce the 'classical' ones (23)):

$$\tilde{L}_q(n,k)=\frac{n!}{k!}[n-1;k-1] \qquad L_q(n,k)=q^{\binom{k}{2}}\tilde{L}_q(n,k).$$

Due to the mixing of 'ordinary' and q-numbers there does not seem to exist a sufficiently simple recursion relation for these generalized Lah numbers (as was already noted in [13]). As an example, one has $\tilde{L}_q(n+1,k)=\tilde{L}_q(n,k-1)+\{(k-1)+q^{k-1}(n+1)\}\tilde{L}_q(n,k)$. Since (51) was derived in complete analogy with the undeformed case (22) using the normal ordering of certain operators, one is led to the conclusion that—at least in the physical context—$L(n,k|q)$ is the 'more natural' q-deformed Lah number. Let us consider the general case where $r=s$. As the common generalization of (20) and (40) one has

$$\{[x]^{\underline{r}}\}^n=\sum_{k=r}^{rn}S_{r,r}(n,k|q)[x]^{\underline{k}} \tag{53}$$

and the common generalization of (15) and (39) is

$$S_{r,r}(n,k|q)=\frac{(-1)^k}{[k]!}\sum_{p=0}^{k}(-1)^p q^{\binom{k-p}{2}}[k;p]\{[p]^{\underline{r}}\}^n. \tag{54}$$

Now, let us briefly discuss the q-deformed version of the generalized Dobinski relation (26). The starting point is (48) with $r = s$ and where we choose $f(x) = e_q(x)$. Since

$$[x^r(D_q)^r]^n \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \sum_{k=r}^{\infty} \frac{\{[k]^{\underline{r}}\}^n x^k}{[k]!}$$

choosing $x = 1$ allows one to infer from (48) that

$$B_{r,r}(n|q) = \frac{1}{e_q(1)} \sum_{k=r}^{\infty} \frac{\{[k]^{\underline{r}}\}^n}{[k]!}. \tag{55}$$

Clearly, this reduces for $r = 1$ to the usual q-Dobinski relation (35) and for $q = 1$ to the generalized Dobinski relation (26). Using the q-deformed coherent states (44), it is easy to check that for $|z|^2 = 1$ one has

$$\langle q; z|[(a^\dagger)^r a^s]^n|z; q\rangle = B_{r,s}(n|q) \tag{56}$$

generalizing the case $r = s = 1$ of (45). This implies that (still assuming $|z|^2 = 1$)

$$\langle q; z|e_q\{\lambda(a^\dagger)^r a^r\}|z; q\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{[n]!} B_{r,r}(n|q).$$

Now, let us consider the matrix elements $\langle q; l|[(a^\dagger)^r a^s]^n|m; q\rangle$. It is sufficient to consider the case where $l = m + n(r - s)$, since in the other cases the matrix elements vanish. The result is

$$\langle q; m + n(r - s)|[(a^\dagger)^r a^s]^n|m; q\rangle = \sum_{k=s}^{\min(sn,m)} \sqrt{\frac{[m + n(r - s)]!}{[m]!}} \frac{[m]!}{[m - k]!} S_{r,s}(n, k|q).$$

Note that for $r = s$ the square root on the right-hand side reduces to 1. In particular, this gives for $\langle q; m|e_q\{\lambda(a^\dagger)^r a^r\}|m; q\rangle$ an expression nearly identical to (46), except that $\min(n, m)$ gets replaced by $\min(rn, m)$ and $S(n, k|q) \equiv S_{1,1}(n, k|q)$ by $S_{r,r}(n, k|q)$. Considering $q = 1$ yields the corresponding formulae of the undeformed case. Now, let us consider the number operator N of the q-deformed boson introduced in section 4, see (28). Using (43), it is easy to see that $a^\dagger a = [N]$. Although N is not equal to $a^\dagger a$, it can be expressed through creation ($a^\dagger$) and annihilation (a) operators [46, 47] (see also [48]),

$$N = \sum_{r=1}^{\infty} \frac{(1 - q)^r}{1 - q^r} (a^\dagger)^r a^r \equiv \sum_{r=1}^{\infty} \nu_r (a^\dagger)^r a^r. \tag{57}$$

Its mth power is therefore given by

$$N^m = \sum_{r_1,r_2,\ldots,r_m=1}^{\infty} \nu_{r_1} \cdots \nu_{r_m} (a^\dagger)^{r_1} a^{r_1} \cdots (a^\dagger)^{r_m} a^{r_m}. \tag{58}$$

Let us split the sum in a 'diagonal' part where all r_i are equal (i.e., $r_1 = r_2 = \cdots = r_m$) and the rest consisting of those tupels $(r_1, \ldots, r_m) \in R$, where not all r_i are equal. It follows that

$$N^m = \sum_{r=1}^{\infty} \nu_r^m [(a^\dagger)^r a^r]^m + \sum_{(r_1,\ldots,r_m)\in R} \nu_{r_1} \cdots \nu_{r_m} (a^\dagger)^{r_1} a^{r_1} \cdots (a^\dagger)^{r_m} a^{r_m}.$$

Using (56), the matrix elements of N^m with respect to the coherent states (44) can be expressed as (assuming again $|z|^2 = 1$)

$$\langle q; z|N^m|z; q\rangle = \sum_{r=1}^{\infty} \nu_r^m B_{r,r}(m|q) + \sum_{(r_1,\ldots,r_m)\in R} \nu_{r_1} \cdots \nu_{r_m} \langle q; z|(a^\dagger)^{r_1} \cdots a^{r_m}|z; q\rangle.$$

The explicit evaluation of the second sum is a straightforward (but tedious) computation involving repeated application of (41) and will not be done here. Note that considering the matrix elements $\langle q; n|N^m|n; q\rangle$ will involve the q-deformed generalized Stirling numbers $S_{r,r}(m, k|q)$ instead of the Bell numbers $B_{r,r}(m|q)$.

6. Conclusions

Following the approach of Katriel and co-workers [2, 20, 30, 31], we have considered (generalized) Stirling numbers and their q-deformed version by studying closely the commutator relation of bosons and a q-deformed version of it. In particular, we have introduced certain 'fermionic' Stirling numbers as well as a q-deformed version of the generalized Stirling numbers introduced in [32–35] and showed that the simplest example for the latter is given by the q-deformed Lah numbers introduced in [9, 10]. We have shown that the matrix elements of the operators $[(a^\dagger)^r a^s]^n$ with respect to the ususal Fock space basis involve the generalized Stirling numbers, whereas considering the matrix elements with respect to coherent states involve the generalized Bell numbers. It is tempting to generalize (13) further and introduce generalized Stirling numbers of 'rank m' (where those of (13) correspond to $m = 1$ and the 'classical' ones to $m = 1$ where in addition $r_1 = s_1 = 1$). To do this, let $r_1, s_1, r_2, s_2, \ldots, r_m, s_m$ be some natural numbers satisfying $r_i \geqslant s_i$ and let $R = r_1 + \cdots + r_m$ and similarly $S = s_1 + \cdots + s_m$ (note that $R \geqslant S$). Now, define the Stirling numbers of rank m by

$$[(a^\dagger)^{r_1} a^{s_1} (a^\dagger)^{r_2} a^{s_2} \cdots (a^\dagger)^{r_m} a^{s_m}]^n = (a^\dagger)^{n(R-S)} \sum_{k=s_m}^{nS} S_{r_1,s_1,\ldots,r_m,s_m}(n,k)(a^\dagger)^k a^k.$$

Clearly, for $m = 1$ they reproduce the generalized Stirling numbers studied above. Of course, one can analogously define the q-deformed version $S_{r_1,s_1,\ldots,r_m,s_m}(n,k|q)$. These numbers should be studied in the same fashion as those of rank one (e.g., recurrence relations, associated Bell numbers). Note that one can now rewrite (58) as

$$N^m = \sum_{r_1,r_2,\ldots,r_m=1}^{\infty} \sum_{k=r_m}^{R} \nu_{r_1} \cdots \nu_{r_m} S_{r_1,r_1,\ldots,r_m,r_m}(1,k|q)(a^\dagger)^k a^k.$$

As a very simple example consider $(a^\dagger a a^\dagger a)^n = \sum_{k=1}^{2n} S_{1,1,1,1}(n,k)(a^\dagger)^k a^k$. Since one may first use the commutation relation inside the bracket yielding $[(a^\dagger)^2 a^2 + a^\dagger a]^n$, application of the binomial theorem and the definition of the Stirling numbers of rank one yields after some algebra for the Stirling numbers $S_{1,1,1,1}(n,k)$ of rank two

$$S_{1,1,1,1}(n,k) = \sum_{r=0}^{n} \sum_{l=2}^{2r} \sum_{m=1}^{n-r} \binom{n}{r} \binom{l}{k-m} \frac{m!}{(k-l)!} S_{2,2}(r,l) S_{1,1}(n-r,m)$$

(with the obvious conventions). Note that in the general case the number of different orders of application of the commutation relations and the definition of Stirling numbers of various ranks will yield a huge amount of identities among Stirling numbers of various ranks. As a final point we want to mention that a connection between the (p,q)-deformed oscillator introduced in [49] and studied further in, e.g., [50, 51], and the (p,q)-deformed Stirling numbers of [11] was already found in [20]. However, it seems that no attempt has been made to study this connection closer (studying, e.g., associated Bell numbers, Dobinski relations and the connection to coherent states), although it obviously deserves more interest.

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